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On the Upper Monophonic Global Domination Number of a Graph

V. Selvi and *V. Sujin Flower

Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 4
Affiliated to Manonmaniam Sundaranar University, Tirunelveli - 627012
*Corresponding Author - Email: sujinflower@gmail.com

ABSTRACT

A monophonic global dominating set M *of* G *is called a minimal monophonic global dominating set of* G *if no proper subset of* M *is a monophonic global dominating set of* G. *The maximum cardinality of a minimal monophonic global dominating set of* G *is the upper monophonic global dominating set of* G, *denoted by* $\bar{\gamma}_m^+(G)$. *This concept's general qualities are investigated. The upper monophonic global domination number of some family of graphs is determined. It is shown that for any positive integers a and b with* $2 \leq a \leq b$, *there exists a connected graph* G *such that* $\gamma_m(G) = a$ *and* $\bar{\gamma}_m^+(G) = b$. *where* $\gamma_m(G) = a$ *is the monophonic global domination number of* G.

Keywords: *Domination number, Global domination number, Monophonic number, Monophonic global domination number, Upper monophonic global domination number.*

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1. Introduction

A finite, undirected connected graph with no loops or many edges is referred to as a graph $G = (V, E)$ The letters n and m respectively, stand for the order and size of G. Basic graph theoretic terms are taken from [1]. If uv is an edge of G then two vertices u and v are said to be adjacent. If $uv \in E(G)$, then u is $v's$ neighbour and the set of $v's$ neighbours are denoted by $N(v)$ and degree of $v \in V$ has degree $\deg(v) = |N(v)|$. If $\deg(v) = n - 1$, a vertex v is referred to as a universal vertex. A vertex v is called an extreme vertex if $G[N(v)]$ is complete. The subgraph induced by a set S of vertices of a graph G is denoted by $G[S]$ with $V(G[S]) = s$ and $E(G[S]) = \{uv \in E(G): u, v \in S\}$. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v.

A chord of a path P is an edge which connects two non-adjacent vertices of P. An $u - v$ path is called a monophonic path if it is a chordless path. The monophonic distance $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. An $u - v$

monophonic path of length $d_m(u,v)$ is called a $u-v$ monophonic. The monophonic eccentricity $e_m(v)$ of a vertex v in G is the maximum monophonic distance from v and a vertex of G, (i.e). $e_m(v) = max\{d_m(v,u) : u \in V\}$. The minimum monophonic eccentricity among the vertices of G is the monophonic radius, $rad_m G$ and the maximum monophonic eccentricity is its monophonic diameter, $diam_m G$. We denote $rad_m(G)$ by r_m and $diam_m G$ by d_m. Two vertices u and v of G are monophonic antipodal vertex if $d_m(u,v) = d_m$. A vertex v is called a monophonic peripheral vertex of G, if $e_m(v) = d_m$. The monophonic distance of a connected graph was studied by Santhakumaran [2]. For two vertices u and v, the closed interval $J[u,v]$ consists of all the vertices lying in a $u-v$ monophonic path including the vertices u and v. If u and v are adjacent, then $J[u,v] = \{u,v\}$. For a set M of vertices, let $J[M] = \cup_{u,v\in M} J[u,v]$. Then certainly $M \subseteq J[M]$. A set $M \subseteq V(G)$ is called a monophonic set of G if $J[M] = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is called a m-set of G. The monophonic number of a graph was studied in [3,4,5,6].

A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex $u \in S$. The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied by Haynes, Hedetniemi and Slater [7]. A subset $D \subseteq V$ is called a global dominating set in G if D is a dominating set of both G and $\bar{G}$. The global domination number $\bar{\gamma}(G)$ is the minimum cardinality of a global dominating sets in G. The concept of global domination in graph was introduced by Sampathkumar [8] and Vaidya and Pandit haave studies global domination and its properties [9, 10]. A set $M \subseteq V$ is said to be a monophonic global dominating set of G if M is both a monophonic set and a global dominating set of G. The minimum cardinality of a monophonic global dominating set of G is the monophonic global domination number of G and is denoted by $\bar{\gamma}_m(G)$. A monophonic global dominating set of cardinality $\bar{\gamma}_m(G)$ is called a $\bar{\gamma}_m$ -set of G. Throughout the paper, G denotes a connected graph at least two vertices. The following theorem is used in the sequel.

Theorem 1.1. [11] Each extreme vertex of a connected graph G belongs to every monophonic global dominating set of G.

2 The Upper Monophonic Global Domination Number of a Graph

Definition 2.1. A monophonic global dominating set of G is called a minimal monophonic global dominating set of G if no proper subset of M is a monophonic global dominating set of G. The maximum cardinality of a minimal monophonic global dominating set of G is the upper monophonic global dominating set of G, denoted by $\bar{\gamma}_m^+(G)$.

Example 2.2. For the graph G given in Figure 2.1, $M_1 = \{v_1, v_2, v_3\}, M_2 = \{v_1, v_3, v_4\}$, $M_3 = \{v_1, v_3, v_5\}, M_4 = \{v_1, v_3, v_6\}, M_5 = \{v_1, v_5, v_6\}$, $M_6 = \{v_3, v_5, v_6\}$ and $M_7 = \{v_2, v_4, v_5, v_6\}$ are the only seven minimal monophonic global dominating sets of G so that $\bar{\gamma}_m^+(G) \geq 4$. It is easily verified that no 5-element subset of V is a minimal monophonic global dominating set of G, and thus $\bar{\gamma}_m^+(G)$= 4.

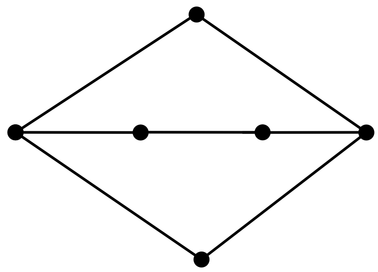

Figure 2.1

Observation 2.3. Let G be a connected graph of order $n \geq 2$. Then

(i) Each extreme vertex of a graph G belongs to every minimal monophonic global dominating set of G. In particular, each end-vertex of G belongs to every minimal monophonic global dominating set of G.

(ii) Each universal vertex of G belongs to every minimal monophonic global dominating set of G.

(iii) Let G be a connected graph and v a cut vertex of G. If M is a minimal monophonic global dominating set of G, then every component of $G - v$ contains an element of M.

(iv) For a connected graph G order $n \geq 2, 2 \leq \bar{\gamma}_m(G) \leq \bar{\gamma}_m^+(G) \leq n$.

(v) For a connected graph G order $n \geq 2, \bar{\gamma}_m(G) = n$ if and only if $\bar{\gamma}_m^+(G) = n$.

(vi) For the star $G = K_{1,n-1}$ $(n \geq 3)$, $\bar{\gamma}_m(G) = \bar{\gamma}_m^+(G) = n$.

(vii) For the complete graph $G = K_n$, $\bar{\gamma}_m(G) = \bar{\gamma}_m^+(G)$ = n.

Theorem 2.4. Let G be a connected graph of order n $\geq$ 2. If $\bar{\gamma}_m^+(G)$= n, then $\bar{\gamma}_m(G)$ = n.

Proof: This follows from observation 2.3 (iv) and (v).

Remark 2.5. The converse of Theorem 2.4 need not be true. For the graph G given in Figure 2.2, $M_1 = \{v_1, v_2, v_3\}$ is a $\bar{\gamma}_m$ - set of G and $M_2 = \{v_1, v_2, v_4, v_5\}$ is a minimal monophonic global dominating set of G so that $\bar{\gamma}_m(G)$ = 3 and $\bar{\gamma}_m^+(G)$= 4.

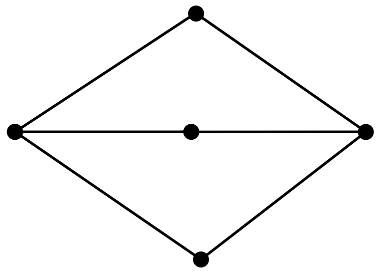

Figure 2.2

Theorem 2.6. For the path $G = P_n$, $(n \geq 4)$, $\bar{\gamma}_m^+(G) = \lceil\frac{n}{2}\rceil$.

Proof: Let $V(P_n) = \{v_1, v_2, .., v_n\}$ and $E(P_n) = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$. We prove this theorem by considering two cases.

Case 1: n is even.

Let $n = 2k$ $(k \geq 2)$. Let $M = \{v_1, v_3, v_5, .., v_{2k-2}, v_{2k}\}$. Then M is a minimal monophonic global dominating set of G and so $\bar{\gamma}_m^+(G) \geq k = \frac{n}{2}$. We prove that $\bar{\gamma}_m^+(G) = \frac{n}{2}$. Suppose this is not the case. Then there exists a minimal monophonic global dominating set M' such that $|M'| \geq \frac{n}{2} + 1$. Hence there exists $v_l v_{l+1} \in M'$, where $1 \leq l \leq n-1$. By Observation 2.3 (i), $v_1, v_n \in M'$. Without loss of generality, let us assume that $v_1 \neq v_l$ and $v_l \neq v_n$. Then $M' - \{v_l\}$ is a monophonic global dominating set of G, which is a contradiction to M'a minimal monophonic global dominating set of G. Therefore $\bar{\gamma}_m^+(G) = \lceil\frac{n}{2}\rceil$.

Case 2: n is odd.

Let $n = 2k + 1$. Let $M = \{v_1, v_3, v_5, .., v_{2k-1}, v_{2k+1}\}$ is a minimal monophonic global dominating set of G and so $\bar{\gamma}_m^+(G) \geq \lceil\frac{n}{2}\rceil$. Using similar argument as in Case (i), we prove that $\bar{\gamma}_m^+(G) = \lceil\frac{n}{2}\rceil$.

Theorem 2.7. For the cycle $G = C_n$ $(n \geq 4)$, $\bar{\gamma}_m^+(G) = \lceil\frac{n}{2}\rceil$.

Proof: The proof is similar to the proof of Theorem 2.6.

Theorem 2.8. For the fan graph $G = K_1 + P_{n-1}$ $(n \geq 4)$, $\bar{\gamma}_m^+(G) = 4$.

Proof: If $n = 4$, then the result follows from Observation 2.3 (i) and (ii). So, let $n \geq 5$. Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, .., v_{n-1}\}$. Let $M = \{x, v_1, v_{n-1}, y\}$, where $y \in \{v_2, v_3, \dots, v_{n-2}\}$. Then M is a minimal monophonic global dominating set of G and so $\bar{\gamma}_m^+(G) \geq 4$. We prove that $\bar{\gamma}_m^+(G) = 4$. Suppose this is not the case. Then there exists a minimal monophonic global dominating set M' such that $|M'| \geq 5$. By Observation 2.3(i) and (ii) $x, v_1, v_{n-1} \in M'$. Hence it follows that $\subset M'$, which is a contradiction to M' a minimal monophonic global dominating set of G, Therefore $\bar{\gamma}_m^+(G) = 4$.

Theorem 2.9. For the complete bipartite $G = K_{r,s}$ $(1 \leq r \leq s)$,

$$\bar{\gamma}_m^+(G) = \begin{cases} r + s, & \text{if } r = 1,\ s \geq 1 \\ s + 1, & \text{otherwise} \end{cases}$$

Proof: Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the two bipartite sets of G.

Case 1: If $r = 1, s \geq 1$. This follows from Observation 2.3(i).

Case 2: $2 \leq r < s$.

Let $M = W \cup \{x\}$, where $x \in U$. Then M is a monophonic global dominating set of G. We prove that M is a minimal global dominating set of G. Suppose this is not the case. Then there exists a monophonic global dominating set M' such that $M' \subset M$.

Let v be a vertex of G such that $v \in M$ and $v \in M'$. If $v = x$, then M' is not a global dominating set of G. If $v = w_i$ for some i $(1 \leq i \leq s)$, then $v \notin J[M']$ and so M' is not a monophonic set of G, which is a contradiction. Therefore M is a minimal global dominating set of G and so $\bar{\gamma}_m^+(G) \geq s + 1$. Note that $M_1 = U \cup \{y\}$, where $y \in W$ and $M_{ijkl} = \{u_i, u_j, w_k, w_l\}(1 \leq i \leq j \leq r)$ $(1 \leq k \leq l \leq s)$ are the minimal monophonic global dominating set of G. We prove that $\bar{\gamma}_m^+(G) = s + 1$. Suppose this is note the case. Then there exists a minimal monophonic global dominating set S such that $|S| \geq s + 2$. Then $S \subset U \cup W$. Since $|S| \geq s + 2$, either $M_1 \subset S$ or $M_{ijkl} \subset S$ for $(1 \leq i \leq j \leq r), (1 \leq k \leq l \leq s)$, which is a contradiction to S a minimal monophonic global dominating set of G. Therefore $\bar{\gamma}_m^+(G) = s + 1$.

Case 3: $r = s$. Using similar argument are as Case 2, we show that $\bar{\gamma}_m^+(G) = r$.

Theorem 2.10. For the wheel graph $G = K_1 + C_{n-1}$, $(n \geq 4)$, $\bar{\gamma}_m^+(G) = 4$.

Proof: Let $V(K_1) = \{x\}$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. Let $M = \{x, u, v, w\}$, where $u, v, w \in V(C_{n-1})$. Then M is a minimal monophonic global dominating set of G and so $\bar{\gamma}_m^+(G) \geq 4$. We prove that $\bar{\gamma}_m^+(G) = 4$. Suppose this is not the case. Then there exists a minimal monophonic global dominating set M' such that $|M'| \geq 5$. By Observation 2.3(ii), $x \in M'$. Since $M' - \{x\} \subset V(C_{n-1})$, We have $M \subset M'$, which is a contradiction to M' a minimal monophonic global dominating set of G. Therefore $\bar{\gamma}_m^+(G) = 4$.

Theorem 2.11. For any positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\bar{\gamma}_m(G) = a$ and $\bar{\gamma}_m^+(G) = b$.

Proof: If $a = b$, let $G = K_{1,a-1}$. Then by Observation 2.3 (vi), $\bar{\gamma}_m(G) = \bar{\gamma}_m^+(G) = a$. So, let

$2 \leq a < b$. Let $V(K_2) = \{x, y\}$ and $V(K_{b-a+1}) = \{u_1, u_2, \ldots, u_{b-a+1}\}$. Let $H = K_{b-a+1} + K_2$. Let G be the graph in Figure 2.3 obtained from H by adding $a-1$ new vertices $z_1, z_2, \ldots, z_{a-1}$ and joining each vertex z_i $(1 \leq i \leq a-1)$ with y. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$. By Observation 2.3 (i), Z is a subset of every monophonic global dominating set of G. Since $J[Z] \neq V$, Z is not a monophonic global dominating set of G and so $\bar{\gamma}_m(G) \geq a$. Let $M = Z \cup \{x\}$. Then M is a monophonic global dominating set of G so that $\bar{\gamma}_m(G) = a$.

Next, we prove that $\bar{\gamma}_m^+(G) = b$. Let $T = Z \cup \{u_1, u_2, \ldots, u_{b-a+1}\}$. Then T is is a monophonic global dominating set of G. We show that T is a minimal monophonic global dominating set of G. Let W be any proper subset of T. Then there exist at least one vertex say $v \in T$ such that $v \notin W$. By Observation 2.3 (i), $v \neq z_i$ for all i $(1 \leq i \leq a-1)$. Now, assume that $v = u_j$ for some j $(1 \leq j \leq b-a+1)$. Then $u_i \notin J[W]$ and so W is not a monophonic global dominating set of G. Hence T is a minimal monophonic global dominating set of G so that $\bar{\gamma}_m^+(G) \geq$ b.

Now, we show that there is no minimal monophonic global dominating set S of G with $|S| \geq b+1$. Suppose that there exists a minimal monophonic global dominating set S of G such that $|S| \geq b+1$. Since $|V(G)| = b+2$ and since M is a monophonic global dominating set of G, it follows that $|S| = b+1$. It is easily seen that, $y \notin S$ and so $S = V(G) - \{y\}$. Since M is a monophonic global dominating set of G, it follows that S is not a minimal a monophonic global dominating set of G, which is a contradiction. Thus $\bar{\gamma}_m^+(G) = b$.

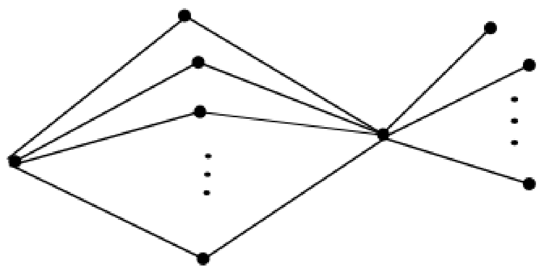

Figure 2.3

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